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On calculating the potential vorticity flux

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We discuss and compare different approaches to calculating the dynamics of anisotropic flow structure formation in quasi two-dimensional turbulence based on potential vorticity (PV) transport in real space. The general structure of the PV flux in the relaxation processes is deduced *non-perturbatively*. The transport coefficients of the PV flux are then systematically calculated using perturbation theory. We develop two non-perturbative relaxation models: the first is a mean field theory for the dynamics of minimum enstrophy relaxation based on the requirement that the mean flux of PV dissipates total potential enstrophy but conserves total fluid kinetic energy. The results show that the structure of PV flux has the form of a sum of a positive definite hyper-viscous and a negative or positive viscous flux of PV. Turbulence spreading is shown to be related to PV mixing via the link of turbulence energy flux to PV flux. In the relaxed state, the ratio of the PV gradient to zonal flow velocity is homogenized. This homogenized quantity sets a constraint on the amplitudes of PV and zonal flow in the relaxed state. The second relaxation model is derived from symmetry principles alone. The form of PV flux contains a nonlinear convective term in addition to viscous and hyper-viscous terms. For both cases, the transport coefficients are calculated using perturbation theory. For a broad turbulence spectrum, a modulational calculation of the PV flux gives both a negative viscosity and a positive hyper-viscosity. For a narrow turbulence spectrum, the result of a parametric instability analysis shows that PV transport is also convective. In both relaxation and perturbative analyses, it is shown that turbulent PV transport is sensitive to flow structure, and the transport coefficients are nonlinear functions of flow shear.
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I. INTRODUCTION

The formation of large-scale shearing structures due to momentum transport—i.e., zonal flow formation—is a common feature of both geostrophic fluids and magnetically confined plasmas (e.g., Refs. 1–5). We study the dynamics of structure formation from the perspective of potential vorticity (PV) transport in real space. The reason that PV mixing is the key element of zonal flow formation is that PV conservation is the fundamental freezing-in law constraint on zonal flow generation by inhomogeneous PV mixing. Note that since zonal flows are elongated, asymmetric vortex modes, translation symmetry in the direction of the flow and inhomogeneity across the direction of the flow are essential to zonal flow formation. The importance of PV mixing to the zonal flow problem is clearly seen via the Taylor identity, which states that the cross-flow flux of PV equals the along-flow component of the Reynolds force, which drives the flow. PV mixing is related to disparate-scale interaction between two classes of fluctuations, namely, turbulence and waves, and zonal flows. Most of the theoretical calculations of PV flux for zonal flow generation are perturbative analyses and focus on the stability of ambient wave spectrum to a seed zonal flow.[6,7] These types of analyses are, however, valid only in the initial stage of zonal flow formation. Therefore, there is a need to develop a mean field theory based on general, structural principles, and not limited by perturbative methods. Here, we examine and compare two approaches to the question of how to obtain the general form of the PV flux: the selective decay hypothesis and the joint reflection symmetry principle.

In the first approach, we study the dynamics of the PV flux during a selective decay process toward a minimum enstrophy state. The relaxed state of a high Reynolds number, turbulent, two-dimensional fluid is thought to be one of minimum potential enstrophy, for given conserved kinetic energy. This hypothesis constitutes the minimum enstrophy principle of Bretherton and Haidvogel.[8] Their variational argument is based on the concept of selective decay, which is in turn based on the dual cascade[9] in two-dimensional turbulence. In two-dimensional turbulence, kinetic energy inverse cascades to large, weakly dissipated spatial scales, whereas enstrophy forward cascades to small spatial scales and there is viscously damped. In the presence of weak dissipation, total kinetic energy is thus approximately conserved relative to total enstrophy, which is dissipated. In this scenario, the system evolves toward a state of a minimum enstrophy. Interestingly, the theory does not specify that the minimum enstrophy is actually achieved in the relaxed state. The theory predicts the structure of the flow in the end state; however, it gives no insight into the all-important question of how the mean profiles evolve during the relaxation process. Here, we discuss the *dynamics* of minimum enstrophy relaxation, which leads to zonal flow formation. In particular, since inhomogeneous PV mixing is the fundamental mechanism of zonal flow formation, we ask *what form must the* *mean field PV flux have so as to dissipate enstrophy while conserving energy*?

We derive a mean field theory for the PV flux during minimum enstrophy relaxation. We show that the structure of the PV flux which dissipates enstrophy is not Fickian diffusion of PV; rather it is $\Gamma_q = \langle v_x \rangle^{-1} \nabla[\mu \nabla(\nabla \langle q \rangle / \langle v_x \rangle)]$, where q is PV and the proportionality coefficient μ is a function of zonal velocity. In other words, PV flux is a form involving viscosity and hyper-viscosity, with flow-dependent transport coefficients. Among the possible forms of PV flux which can minimize enstrophy while conserving energy, we consider the simplest, smoothest solution in this paper. *We show that in the relaxed state, the ratio between PV gradient and zonal flow is homogenized.* Interestingly, this proportionality relationship between PV gradient and zonal flow is observed in PV staircases.

Turbulence spreading[10–12] is closely related to PV mixing because the transport of turbulence intensity, namely, fluctuation energy or potential enstrophy, has influence on Reynolds stresses and flow dynamics. The momentum theorems for the zonal flow in Rossby/drift wave turbulence[13] link turbulent flux of potential enstrophy density to zonal flow momentum and turbulence pseudomomentum, along with the driving flux and dissipation. Here, note that the pseudomomentum, or wave momentum density, is defined as $-\langle \tilde{q}^2 \rangle / 2(\partial_y \langle q \rangle)$ for a quasi-geostrophic system, and so is proportional to the wave action density in the weakly nonlinear limit. In this work, turbulence spreading is linked to PV mixing via the relation of energy flux to PV flux. The turbulent flux of kinetic energy density during minimum enstrophy relaxation is shown to be proportional to the gradient of the (ultimately homogenized) quantity, which is the ratio of PV gradient to the zonal flow. A possible explanation of up-gradient transport of PV due to turbulence spreading—which is based on the connection between PV mixing and turbulence spreading—is discussed in Sec. IV.

The structural approach of the minimum enstrophy relaxation model exploits ideas from the study of relaxation dynamics in three-dimensional magnetohydrodynamic (MHD), given the analogy between minimum enstrophy relaxation in two-dimensional turbulence and Taylor relaxation[14] in three-dimensional MHD turbulence. In Taylor relaxation, magnetic energy is minimized subject to the constraint of conservation of global magnetic helicity. Taylor's conjecture is based on the concept of selective decay and the assumption of magnetic field line stochasticity during turbulent relaxation. In 3D MHD turbulence, energy forward cascades to small scales, while magnetic helicity inverse cascades to large scales. Thus, energy is dissipated, while magnetic helicity is rugged. The flux tube which is most rugged on the longest time scales is the tube which contains the entire system, resulting from the stochasticity of field lines. The relaxed state of the Taylor process is a force free magnetic field configuration. The parallel current profile in the Taylor state is homogenized, so there is no available free energy in the current gradient. The related criteria for stability of the magnetostatic equilibrium of an arbitrarily prescribed topology during magnetic relaxation in a perfectly conducting fluid are further discussed by Moffatt.[15] The Taylor hypothesis is successful in predicting the magnetic field configuration of some laboratory plasmas and astrophysical plasmas. However, as for the minimum enstrophy principle, Taylor relaxation theory does not address the dynamics of the relaxation, which is characterized by the helicity density flux. Boozer[16] argues that the simplest form of helicity density flux which dissipates magnetic energy is that of diffusion of current or "hyper-resistivity"—hyper-diffusion of magnetic helicity. The dynamical model of Taylor relaxation and helicity transport was developed further by Diamond and Malkov[17] using more general symmetry considerations than Boozer's. In particular, they use the joint reflection symmetry principle[18] to show that the current profile on mesoscales evolves according to a Burgers equation, suggesting that the helicity transport is non-diffusive and intermittent during Taylor relaxation. In particular, a $1/f$ spectrum of helicity "transport events" or "avalanches" is predicted.

In the second approach, we follow Diamond and Malkov and derive a simple "PV-avalanche" model of the dynamics of turbulent relaxation of the excursion from the self-organized profile using symmetry principles alone. We ask *what form must the form of the PV flux have so as to satisfy the joint reflection symmetry principle?* The result is a sum of a viscous, a hyper-viscous, and a convective transport of PV. The PV equation has the same structure as the Kuramoto-Sivashinsky equation, which is known for its negative diffusion (large-scale instability) and higher-order stabilizing diffusion (small-scale damping). Comparing the minimum enstrophy model and the PV-avalanche model, we find that the structure of viscous and hyper-viscous transport of PV appears in both models, while the convective transport of PV, which suggests intermittent PV transport during turbulence self-organization, is only found in the PV-avalanche analysis. Nevertheless, we note that the nonlinear convective term of the PV flux can be viewed as a generalized diffusion, on account of the gradient-dependent ballistic transport in avalanche-like systems.

We note that the minimum enstrophy state may not be precisely the same as the self-organized state, on account of the dissipation, external drive, and boundary conditions of the system. We also note that selective decay is a *hypothesis* based on the observation of the dual cascade in two-dimensional turbulence, and is not derived from first physical principles. There are relaxed states derived from more fundamental principles, namely, statistical equilibrium states and stable stationary states (see, e.g., Refs. 19–23). However, the minimum enstrophy principle is a plausible and demonstrably useful guide, which gives us predictions of the structure of PV and flows, and the enstrophy level in the relaxed state. The selective decay principles can and have been applied in a number of areas of physics, such as MHD and geophysics. Selective decay hypotheses have been supported by a number of computational studies (e.g., Refs. 24 and 25) and experimental studies (e.g., successful prediction of the magnetic configuration of reversed field pinch plasmas). The minimum enstrophy state is an attracting state when the system evolves freely, i.e., when there is no external forcing. The final state is determined by the balance between free relaxation and forcing. Determining the exact ultimate state for any particular system is an extremely difficult question and is beyond the scope of this work.

To systematically study the dynamics of PV flux, we calculate the transport coefficients using perturbation theory in two cases: (1) modulational instability for a broad turbulence spectrum and (2) parametric instability for a narrow turbulence spectrum. The results of modulational calculation show that the PV flux contains a negative viscosity and a positive hyper-viscosity. The viscous and the hyper-viscous transport of PV is found in both the minimum enstrophy relaxation model and the PV-avalanche model. For parametric instability, the linear growth rate shows that turbulent vorticity transport is convective. *The results of both relaxation principles and perturbative analyses show that PV flux is not well represented by Fickian diffusion*. While considerable progress has been made in the wave-mean interaction theory of zonal flow generation and the flow structure in the end relaxed state; here, we present a study of the dynamics of PV transport which links these two questions.

The rest of the paper is organized as follows. Section II presents the non-perturbative analyses of PV flux, including the use of the minimum enstrophy principle in Sec. II A, and the joint reflection symmetry principle in Sec. II B. Section III derives momentum transport coefficients via perturbation theory, including modulational instability in Sec. III A and parametric instability in Sec. III B. The discussion, synthesis, and conclusions are given in Sec. IV. The structure of this paper is illustrated in the flowchart of Fig. 1.

II. DEDUCING THE FORM OF THE PV FLUX FROM NON-PERTURBATIVE ANALYSES

A. Minimum enstrophy principle

We approach the question of the dynamics of momentum transport in 2D turbulence by asking *what the form of PV flux must be to dissipate enstrophy but conserve energy.* We start with the conservative PV evolution equation

$$\partial_t q + v \cdot \nabla q = \nu_0 \nabla^2 q, \tag{1}$$

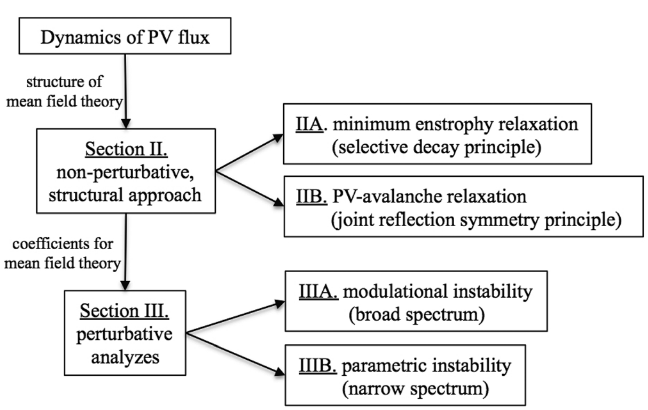

FIG. 1. Flowchart for the paper.

where ν_0 is molecular viscosity. Equation (1) states PV as a material invariant and so applies to many quasi-2D systems, including, but not limited to, the following two systems. In 2D quasigeostrophic turbulence,[2] the PV and velocity fields are $q = \nabla^2\psi + \beta y$ and $(v_x, v_y) = (-\partial\psi/\partial y, \partial\psi/\partial x)$, where ψ is the streamfunction and β is the latitudinal gradient of the Coriolis parameter. In drift wave turbulence,[26] the PV is $q = n - \nabla^2\phi$, where n is the ion density and the Laplacian of the electrostatic potential ϕ is the ion vorticity due to $\boldsymbol{E} \times \boldsymbol{B}$ drift. In this paper, we use the coordinates of a 2D geostrophic system; the x-axis is in the zonal direction, the direction of symmetry (the poloidal direction in tokamaks), and the y-axis is in the meridional direction, the direction of anisotropy (the radial direction in tokamaks). Periodic boundary conditions in the $\hat{x}$ direction are imposed. Writing the nonlinear term, which is the divergence of the PV flux, as N

$$N \equiv -\nabla \cdot (\boldsymbol{v} q), \tag{2}$$

we average Eq. (1) over the zonal direction to get the mean field equation for PV

$$\partial_t \langle q \rangle = \langle N \rangle + \nu_0 \partial_y^2 \langle q \rangle. \tag{3}$$

The selective decay hypothesis states that 2D turbulence relaxes to a minimum enstrophy state. During relaxation, the enstrophy forward cascades to smaller and smaller scales until it is dissipated by viscosity. Thus, the total potential enstrophy

$$\Omega = \frac{1}{2}\int q^2\, dxdy, \tag{4}$$

will decrease with time. On the other hand, the kinetic energy inverse cascades to large scales and sees negligible or weak coupling to viscous dissipation, as compared to the enstrophy. Only scale invariant frictional drag can damp flow energy at large scales. The rate of large scale energy drag is much slower than the rate of small scale enstrophy dissipation. Thus, the total kinetic energy

$$E = \frac{1}{2}\int (\nabla\psi)^2\, dxdy, \tag{5}$$

should remain invariant on the characteristic enstrophy dissipation time. Note that the total energy of some systems consists of kinetic, potential, and internal energies, but only the kinetic energy is conserved in the minimum enstrophy hypothesis. This is because the nonadiabatic internal energy (i.e., $\sim\langle(\tilde{n}/n - e\tilde{\phi}/T)^2\rangle$ for drift wave turbulence) forward cascades to dissipation.[27] The evolution of the total kinetic energy is given by

$$\partial_t E = -\int \psi \partial_t (\partial_x^2 + \partial_y^2)\psi\, dxdy + \int \partial_x(\psi\partial_t\partial_x\psi)dxdy + \int \partial_y(\psi\partial_t\partial_y\psi)dxdy. \tag{6}$$

The second term of Eq. (6) vanishes because of the periodic boundary condition in $\hat{x}$ direction, and the third term is dropped due to the condition of zero stream function or zero zonal flow at the $\pm y_0$ boundaries, $\langle\psi\partial_y\psi\rangle|_{\pm y_0}=0$. Therefore, conservation of the total kinetic energy (apart from feeble collisional dissipation) in mean field theory gives

$$\partial_t E = -\int\langle\psi\rangle\partial_t\langle q\rangle\, dxdy = -\int\langle\psi\rangle\langle N\rangle\, dxdy = -\int\partial_y\Gamma_E dxdy, \tag{7}$$

where $\langle\,\rangle$ is the zonal average, and the energy density flux Γ_E is defined as $\langle v_y\frac{(\nabla\psi)^2}{2}\rangle$. Thus, the nonlinear term is necessarily tied to the energy flux by

$$\langle N\rangle = \langle\psi\rangle^{-1}\partial_y\Gamma_E. \tag{8}$$

The form of the energy density flux is constrained by the requirement of decay of total potential enstrophy, i.e., by the demand that

$$\partial_t\Omega = \int\langle q\rangle\langle N\rangle dxdy = \int\langle q\rangle\langle\psi\rangle^{-1}\partial_y\Gamma_E\, dxdy < 0. \tag{9}$$

Noting that the energy flux vanishes at the $\pm y_0$ boundaries, (i.e., $\Gamma_E|_{\pm y_0}=0$), and so Eq. (9) becomes

$$\partial_t\Omega = -\int\Gamma_E\,\partial_y(\langle q\rangle\langle\psi\rangle^{-1})dxdy < 0, \tag{10}$$

which in turn forces

$$\Gamma_E = \nu\partial_y(\langle q\rangle\langle\psi\rangle^{-1}). \tag{11}$$

It is worthwhile mentioning here that a finite flux at the boundary would contribute a surface integral term to the total enstrophy evolution. PV relaxation at the point y would then become explicitly dependent upon fluxes at the boundary, thus rendering the mean field theory manifestly non-local. We therefore see this as an important topic for future research. The simplest solution for Γ_E is for it to be directly proportional to $\partial_y(\langle q\rangle\langle\psi\rangle^{-1})$, with a positive proportionality parameter ν. The obvious requisite dependence on turbulence intensity is contained in ν. In this mean field theory, ν is not determined. Note that any odd derivate of $\langle q\rangle\langle\psi\rangle^{-1}$ or any combination of an odd derivative of $\langle q\rangle\langle\psi\rangle^{-1}$ and an even power of $\langle q\rangle$ or $\langle v_x\rangle$ will contribute a term which dissipates enstrophy. For example, $\Gamma_E=\nu\langle q\rangle^2\partial_y(\langle q\rangle\langle\psi\rangle^{-1}) +\partial_y^5(\langle q\rangle\langle\psi\rangle^{-1})$ also gives $\partial_t\Omega<0$. Thus, the solution we present here is the smoothest (i.e., dominant in long wavelength limit), and lowest order (i.e., not combined with any higher power of $\langle q\rangle^2$ or $\langle v_x\rangle^2$). The reasons we study the simplest solution are: (1) the smoothest solution reveals the leading behavior of the PV flux on large scale. This is relevant to our concern with the large-scale flow dynamics. The higher order derivatives should be included to study the relaxation dynamics at smaller scales and the finer scale structure of the shear flow. (2) The dependence of PV flux on the higher order powers of the shear flow intensity can be absorbed into ν. The nonlinear term and the PV equation are then given by the simplest, leading form of Γ_E

$$\langle N\rangle = \langle\psi\rangle^{-1}\partial_y[\nu\partial_y(\langle q\rangle\langle\psi\rangle^{-1})], \tag{12}$$

and

$$\partial_t\langle q\rangle = \nu_0\partial_y^2\langle q\rangle + \langle\psi\rangle^{-1}\partial_y[\nu\partial_y(\langle q\rangle\langle\psi\rangle^{-1})]. \tag{13}$$

The system evolves to the relaxed state, $\partial_t\langle q\rangle=0$, when $\langle q\rangle\langle\psi\rangle^{-1}$ approaches a constant, i.e., $\partial_y(\langle q\rangle\langle\psi\rangle^{-1})=0$, where the nonlinear term is annihilated and the mean PV flux vanishes. This is consistent with the results from calculus of variations, in which the enstrophy is minimized at constant energy, so

$$\delta\Omega+\lambda\delta E = \int q\delta(\nabla^2\psi)\, dxdy + \lambda\int\nabla\psi\cdot\nabla\delta\psi\, dxdy = \int(q-\lambda\psi)\nabla^2\delta\psi\, dxdy, \tag{14}$$

is required to vanish. Here, $\langle q\rangle\langle\psi\rangle^{-1}$ is equal to the Lagrange multiplier λ. In the relaxed state, PV is constant along streamlines and so the nonlinear term $v\cdot\nabla q$ is annihilated. A linear relation between vorticity and stream function as a result of eliminating the nonlinear couplings has been predicted by Leith.[28] He predicted the emergence of isolated, stable vortices in two-dimensional flow by minimizing the enstrophy with respect to energy and angular momentum or circulation. Note that his analysis is restricted to axisymmetric equilibria and perturbations. In the presence of inhomogeneity, namely, the β-effect, the energy ultimately inverse cascades to banded flows instead of vortices with circular symmetry. Flierl *et al.*[29] studied exact vortex solutions of the quasi-geostrophic equations and found "modons"—isolated vortex pairs existing in a zonal flow.

Since what we seek is the structure of the PV flux, we prefer to maintain the form of the nonlinear term in the mean PV evolution as an explicit divergence of a PV flux, i.e., now take

$$\langle N\rangle = \langle-\nabla\cdot(\boldsymbol{v}q)\rangle = -\partial_y\Gamma_q, \tag{15}$$

where Γ_q is the PV flux in the direction of inhomogeneity, $\hat{y}$. We repeat the minimum enstrophy analysis as before. Starting with mean field PV equation

$$\partial_t\langle q\rangle = -\partial_y\Gamma_q + \nu_0\partial_y^2\langle q\rangle, \tag{16}$$

conservation of total kinetic energy (with the boundary condition of $\Gamma_q|_{\pm y_0}=0$)

$$\partial_t E = \int\langle\psi\rangle\partial_y\Gamma_q dxdy = -\int\partial_y\langle\psi\rangle\Gamma_q dxdy = -\int\partial_y\Gamma_E dxdy, \tag{17}$$

relates PV flux to energy flux by

$$\Gamma_q = (\partial_y\langle\psi\rangle)^{-1}\partial_y\Gamma_E. \tag{18}$$

The connection between PV flux and energy flux has direct implication for turbulence spreading, which we discuss later in this paper. We then derive the energy flux from the constraint of the dissipation of potential enstrophy (with the boundary condition of $\Gamma_E|_{\pm y_0}=0$)

$$\partial_t \Omega = -\int \langle q \rangle \partial_y \Gamma_q dxdy = -\int \partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle]\Gamma_E\, dxdy < 0, \quad (19)$$

so the simplest, smoothest solution of Γ_E is directly proportional to $\partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle]$, with a positive proportionality parameter μ. We therefore find the form of PV flux to be

$$\Gamma_q = (\partial_y\langle\psi\rangle)^{-1}\partial_y[\mu\partial_y((\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle)] = \langle v_x\rangle^{-1}\partial_y[\mu(\langle v_x\rangle^{-2}\langle q\rangle\partial_y\langle q\rangle + \langle v_x\rangle^{-1}\partial_y^2\langle q\rangle)]. \quad (20)$$

The relaxed state is achieved when $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$ is constant. The difference between the results of N and $\partial_y\Gamma_q$ formulations comes from the treatment of derivatives in the nonlinear term, i.e., taking $N = -\partial_y\Gamma_q$, as we can see clearly from the homogenized quantities in the two approaches, $\langle\psi\rangle^{-1}\langle q\rangle$ and $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$. The derivative of equation $\langle q\rangle = \lambda\langle\psi\rangle$, from the N approach, gives the equation $\partial_y\langle q\rangle = \lambda\partial_y\langle\psi\rangle$, obtained from the Γ_q approach. Thus, the two solutions are consistent. The $\partial_y\Gamma_q$ formulation is more accurate, since it starts with a more precise form of the nonlinear term in PV equation. Γ_q is smoother than N, and hence better satisfies the conditions of the mean-field approximation that the fluctuations around the average value be small, so that terms quadratic in the fluctuations can be neglected. Moreover, while the stream function ψ is unique up to an arbitrary constant, the absolute value of its derivative $\partial_y\psi = -v_x$ has a clear physical meaning. Therefore, we consider the outcome from the $\partial_y\Gamma_q$ approach to be the primary result. The following discussion is based primarily on Eq. (20).

The structure of the PV flux in Eq. (20) contains both hyper-diffusive and diffusive terms. The mean PV evolution

$$\partial_t\langle q\rangle = -\partial_y\left[\frac{1}{\partial_y\langle\psi\rangle}\partial_y\left[\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\psi\rangle}\right)\right]\right] + \nu_0\partial_y^2\langle q\rangle, \quad (21)$$

shows that hyper-viscosity is the leading high k_y dependence and so it controls the small scales. From Eq. (21), we can also prove that hyper-viscosity term damps the energy of the mean zonal flow

$$\partial_t(\partial_y\langle v_x\rangle)^2 = \partial_t\langle q\rangle^2 = -\frac{2\mu}{\partial_y\langle\psi\rangle}\left(\frac{\langle q\rangle\partial_y^4\langle q\rangle}{\partial_y\langle\psi\rangle}\right) + \text{non hyper-viscosity terms}, \quad (22)$$

and

$$-\int\frac{\mu}{\partial_y\langle\psi\rangle}\left(\frac{\langle q\rangle\partial_y^4\langle q\rangle}{\partial_y\langle\psi\rangle}\right)dxdy = -\int\mu\left(\frac{\partial_y^2\langle q\rangle}{\partial_y\langle\psi\rangle}\right)^2 dxdy < 0. \quad (23)$$

Therefore, the hyper-viscosity represents the nonlinear saturation mechanism of zonal flow growth and partially defines the scale dependence of turbulent momentum flux. The other important implication of Eq. (20) is that the PV flux is explicitly zonal flow-dependent. The zonal velocity appears in the denominators of hyper-viscosity and viscosity terms, as well as the diffusion coefficient; this is not seen in perturbative analyses (e.g., Refs. 6 and 7). We emphasize that within the mean field approach; the selective decay analysis for the PV flux in this work is entirely non-perturbative and contains no assumption about turbulence magnitude.

The prediction of the homogenization of $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$ in minimum enstrophy relaxation is a new result. It states explicitly that zonal flows track the PV gradient in the relaxed state, i.e., strong zonal flows are localized to the regions of larger PV gradient. The trend is realized in the PV staircase, where strong jets produced by inhomogeneous PV mixing peak at PV jump discontinuities.[30] The jet pattern of the $\boldsymbol{E}\times\boldsymbol{B}$ staircase is also observed in plasma simulations.[31] Figure 2 shows a cartoon of the PV staircase. Strong zonal flows are located around "the edges of PV steps." We can write the PV gradient as

$$\partial_y\langle q\rangle = \sum_i a_i f(y - y_i), \quad (24)$$

where a_i are constants and f is a function peaked at y_i. f can be approximated as a delta function in the limit of a step profile. Since $\partial_y\langle q\rangle/\langle v_x\rangle$ is a constant, the zonal flow must have the same spatial profile as the PV gradient, i.e.,

$$\langle v_x\rangle = \sum_i b_i f(y - y_i), \quad (25)$$

where $b_i = -\lambda a_i$. While the prediction of the form of the function $f(y - y_i)$ is beyond the scope of this work, the constant proportionality between a_i and b_i reconciles the highly structured profiles of the staircase with the homogenization or mixing process required to produce it. In a related vein, both $\partial_y\langle q\rangle$ and $\partial_y\langle\psi\rangle$ can each be large and variable, though the ratio is constrained. The observation that the structural analysis of selective decay can lead to an end state with a PV staircase-like structure suggests that the staircase may arise naturally as a consequence of minimum enstrophy relaxation. The result also demonstrates the impact of inhomogeneous PV mixing in minimum enstrophy relaxation.

One can define a characteristic scale from the proportionality between PV gradient and zonal flow velocity, i.e.,

$$l_c = \left|\frac{\partial_y\langle q\rangle}{\langle v_x\rangle}\right|^{-1/2}. \quad (26)$$

In minimum enstrophy state, $l_c = |\lambda|^{-1/2}$ and PV flux can vanish on scale l_c. As a result, l_c characterizes the scale at

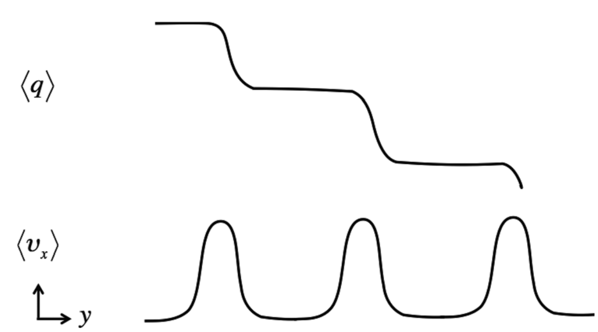

FIG. 2. PV staircase.

which the terms in the PV flux can compete and cancel. For scales smaller than l_c, hyper-viscosity dominates and damping wins. For scales larger than l_c, effective viscosity dominates. Since effective viscosity can be negative, growth can occur for $l > l_c$. It is interesting to compare l_c with the Rhines scale[32] $l_R \sim (\partial_y \langle q \rangle / \widetilde{v}_{rms})^{-1/2}$, where $\widetilde{v}_{rms}$ is the root-mean-square velocity at the energy containing scales. The Rhines scale sets the crossover scale from turbulence-dominated small scales to Rossby/drift wave-dominated large scales, and so determines the scale of zonal flow excitation as a consequence of inverse cascade. The connection between the Rhines scale and zonal flow generation is illustrated in the simulations of geostrophic and drift-wave turbulence models (e.g., Ref. 33). The question of which velocity should really be used to calculate the Rhines Scale is still being debated (see, e.g., Refs. 34 and 35). l_c and l_R both depend on the gradient of the mean field PV; what distinguishes them is that l_c is determined by mean zonal velocity while l_R is set by fluctuation velocity. The characteristic scale and Rhines scale become indistinguishable when $\widetilde{v}_{rms}$ reaches the level of zonal flow velocity.

PV mixing in minimum enstrophy relaxation is also related to turbulence spreading, since we can see from Eq. (18) that Γ_E and Γ_q are related. Γ_E and Γ_q are the spatial flux of kinetic energy density and PV in the direction of inhomogeneity. Since there is no mean flow in the direction of inhomogeneity, Γ_E represents the effective spreading flux of turbulence kinetic energy and is given by

$$\Gamma_E = -\int \Gamma_q \langle v_x \rangle dy = -\int \frac{1}{\langle v_x \rangle} \partial_y \left[\mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right) \right] \langle v_x \rangle dy = \mu \partial_y \left(\frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right). \quad (27)$$

Equation (27) shows that the gradient of the homogenized quantity, $\partial_y (\partial_y \langle q \rangle / \langle v_x \rangle)$, drives spreading, too. Indeed, we see that the spreading flux vanishes when $\partial_y \langle q \rangle / \langle v_x \rangle$ is homogenized. It is known that inhomogeneous turbulence has tendency to relax its intensity gradients through turbulent transport in space. We show that inhomogeneous turbulence during minimum enstrophy relaxation tends to homogenize the gradient of $\partial_y \langle q \rangle / \langle v_x \rangle$ through turbulent transport of momentum (PV mixing) and energy (turbulence spreading). The dependence of Γ_E on zonal flow follows from the fact that turbulence spreading is a mesoscale transport process. Note that the step size of the PV staircase, which corresponds to the distance between zonal flow layers, is also mesoscale. Both observations suggest that the relaxation process is a non-local phenomena. This is a necessary consequence of PV inversion, i.e., the relation $\nabla^2 \psi + \beta y = q$, so that $\langle v_x \rangle$ is an integral of the $q(y)$ profile. Thus, Γ_E and Γ_q are in fact non-local in $q(y)$.

An expression for the relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state. We write $\langle q \rangle = q_m(y) + \delta q(y,t), \langle \psi \rangle = \psi_m(y) + \delta \psi(y,t)$ and use the homogenization condition in relaxed state $\partial_y q_m = \lambda \partial_y \psi_m$. Assuming $\delta q(y,t) = \delta q_0 \exp(-\gamma_{rel} t - i\omega t + iky)$, the relaxation rate is found to be

$$\gamma_{rel} = \mu \left(\frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle v_x \rangle^2} - \frac{8q_m^2 (k^2 + \lambda)}{\langle v_x \rangle^4} \right), \quad \omega = \mu \left(-\frac{4q_m k^3 + 10 q_m k \lambda}{\langle v_x \rangle^3} + \frac{8q_m^3 k}{\langle v_x \rangle^5} \right). \quad (28)$$

The condition of relaxation—i.e., that modes are damped—requires positive γ_{rel}

$$k^2 > \frac{8q_m^2}{\langle v_x \rangle^2} - 3\lambda, \quad (29)$$

i.e., perturbation scales smaller than $(8q_m^2/\langle v_x \rangle^2 + 3\,\partial_y q_m / \langle v_x \rangle)^{-1/2}$. $k^2 > 0$ relates q_m to λ and $\langle v_x \rangle$ by

$$\frac{8q_m^2}{\langle v_x \rangle^2} > 3\lambda. \quad (30)$$

Equation (30) shows that zonal flow cannot grow arbitrarily large and is constrained by the potential enstrophy density and scale parameter λ. It also shows that a critical residual enstrophy density q_m^2 is needed in the minimum enstrophy state, so as to sustain a zonal flow of a certain level. Equation (30) specifies the "minimum enstrophy" of relaxation. Therefore, we not only obtain the structure of the end state, which is expressed in terms of λ, the constant of proportionality between PV gradient and zonal flow velocity, but we also observe that potential enstrophy intensity and zonal flow strength are ultimately related in the relaxed state. It is interesting to note that the PV evolves like a damped oscillator near the relaxed state.

B. Symmetry principles

We have derived the form of PV flux using the minimum enstrophy principle. In this subsection, we look at the problem using more general considerations—namely, flux symmetry principles. We discuss the general form of PV flux near a self-organized state, which is not specified. When the PV profile $q(y)$ deviates from that of the self-organized state $q_0(y)$, the system tends to regulate itself and relax to the self-organized state. Note that the self-organized state may not be precisely the same as the minimum enstrophy state, taking account of the dissipation, external drive, and boundary conditions of the system. We view the relaxation to a self-organized state as similar to relaxation of a running sandpile[18] and consider local PV as analogous to the local sand grain density. Then the deviation of the local PV profile from the self-organized state $\delta q(y) = q(y) - q_0(y)$ drives the PV flux. The dynamics of self-organization is complex. However, the underlying symmetries of the problem allow us to construct a possible general form for the PV flux.

Due to the conservation of PV, δq evolves according to

$$\partial_t \delta q + \partial_y \Gamma[\delta q] = \nu_0 \partial_y^2 \delta q + s, \quad (31)$$

where $\Gamma[\delta q]$ is the flux of δq and s represents the external sources and sinks. We assume that the dynamics of the relaxation process to the self-organized state is similar to the running sandpile models of Hwa and Kardar,[18] Diamond and

Hahm,[36] and Diamond and Malkov,[17] i.e., the excesses beyond the self-organized profile ($\delta q > 0$) move down the local PV gradient while voids ($\delta q < 0$) move up the gradient as illustrated in Fig. 3. Therefore, the form of $\Gamma[\delta q]$ is invariant under $y \to -y$ and $\delta q \to -\delta q$. The general form of $\Gamma[\delta q]$ which satisfies this symmetry constraint is given by

$$\Gamma[\delta q] = \sum_l \alpha_l (\delta q)^{2l} + \sum_m \beta_m (\partial_y \delta q)^m + \sum_n \gamma_n (\partial_y^3 \delta q)^n + \cdots. \tag{32}$$

We are interested in the large-scale properties of the system, so higher-order spatial derivatives are neglected. Assuming the deviations to be small, we also drop the higher-order terms in δq. Thus, the simplest approximation becomes

$$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q, \tag{33}$$

and δq evolves according to

$$\partial_t \delta q + \alpha \delta q \partial_y \delta q + \beta \partial_y^2 \delta q + \gamma \partial_y^4 \delta q = 0. \tag{34}$$

The transport parameters α, β, γ are not determined by this analysis. Similar to the PV flux derived from the minimum enstrophy analysis, the structure of the PV flux in Eq. (33) contains a diffusive term and a hyper-diffusive term. However, Eq. (33) has another piece—a nonlinear convective term. Since the symmetry principle is more general than the minimum enstrophy principle, it is reasonable that the PV flux derived from the former includes but is not limited to term of the PV flux derived from the latter. Note that the avalanche-like transport is in fact triggered by the deviation of the local gradient from the critical gradient. In the absence of a mean gradient, the deviation from the mean profile (δq) will spread out, but its center remains in the same place. Thus, δq implicitly contains information about the local PV gradient. The transport coefficients, which can be functions of δq (while consistent with the symmetry constraints), are related to the mean PV gradient as well. The ballistic propagation of the self-organized turbulent structures is strongly related to a gradient-dependent effective diffusivity ($\Gamma_q \sim -D(\partial_y q)\partial_y q \to -D(\delta q)\delta q$), which is, in turn, related to the convective component of the PV flux in the PV-avalanche model ($\Gamma[\delta q] \sim \delta q^2 \to -D(\delta q)\delta q$, with $D(\delta q) \to D_0 \delta q$).

Equation (34) is very similar to the Kuramoto-Sivashinsky (K-S) equation in one dimension. In K-S equation, α, β, γ are positive constant; as a result, the second-order spatial derivative term is responsible for an instability at large scales (i.e., negative diffusion), the fourth-order derivative term provides damping at small scales, and the non-linear term stabilizes by transferring energy between large and small scales. Since the competitive roles of large-scale flow growth and saturation by diffusive and hyper-diffusive terms have been shown in the minimum enstrophy model, it is plausible to assume that the parameters β and γ are positive in Eq. (34). The nonlinear term has the same form as that in the Burgers equation, suggesting intermittent PV transport during turbulence self-organization. The numerical studies of K-S equation with fixed boundary conditions have shown various types of "shock" patterns (e.g., Ref. 37).

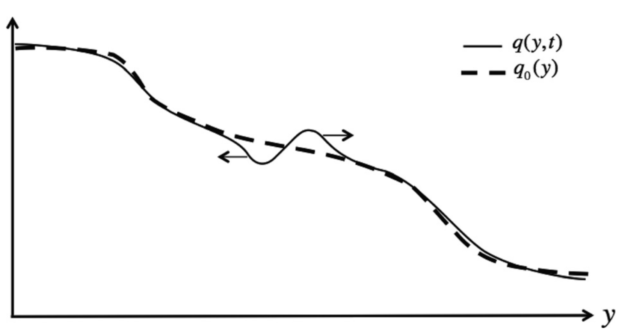

FIG. 3. Positive deviation of the local PV from the self-organized profile q_0 moves down the slope while negative deviation moves up the gradient.

The key issue of the "negative viscosity phenomena" in quasi-geostrophic systems is: what is the form of the momentum/PV flux? While the conventional approaches (e.g., perturbation theory) explore the large scale instabilities, it is interesting to ask whether the diffusion-type flux leading to instabilities is sufficient to describe the whole dynamics. The PV-avalanche model based on general, non-perturbative principles indicates ballistic events, namely, avalanches. A generalized transport coefficient with the dependence on local PV and PV gradient can characterize the negative viscosity phenomena, as well as the ballistic propagation of distortions or "defects" of the PV profiles,[38] which are likely induced by nonlinear wave interactions or external perturbations. Therefore, we suggest that the profile-dependent transport coefficient is, in some sense, a more general representation than the well-known "negative viscosity" for the PV flux associated with jet/structure formation, since PV transport behavior is generically linked to PV profile and its gradient.

III. DEDUCING THE TRANSPORT COEFFICIENTS FROM PERTURBATIVE ANALYSES

In this section, the structure of the PV flux is further analyzed using perturbation theory. The aim is to obtain the turbulent transport coefficients and to study the underlying physics. We study both the modulational instability of a broad fluctuation spectrum and the parametric instability of a narrow fluctuation spectrum, both to a large-scale seed zonal flow. Wave action (population) density conservation is used to evaluate the response of the wave spectrum to the test shear. We consider 2D quasi-geostrophic turbulence ($q = \nabla^2 \psi + \beta y$), in which the wave action density $N_k = \varepsilon_k / \omega_k = -k^4 |\psi_k|^2 / (\beta k_x)$ can be renormalized to the enstrophy density $k^4 |\psi_k|^2$, since β is a constant and k_x is unchanged by zonal flow shearing ($dk_x/dt = -\partial(k_x \langle v_x \rangle)/\partial x = 0$). Thus, we see that the wave action density represents the intensity field of PV and its evolution has a direct connection with PV mixing. The Reynolds force, i.e., PV flux, which drives mean zonal flow is linked to the enstrophy density as $-\partial_y \langle \tilde{v}_y \tilde{v}_x \rangle = \partial_y \int d^2k \frac{k_x k_y}{k^2} (k^4 |\tilde{\psi}_k|^2)$.

While the focus of the perturbative analyses is on modulational and parametric instabilities, we note that there are another approaches to study the interaction between large-scale flow structures and small-scale quasi-geostrophic/drift-wave turbulence such as closure approximations such as the direct interaction approximation (DIA).[39] See in particular, the Markovian quasi-normal (MQN) closure by Holloway and Hendershott,[40] in which a strong zonal anisotropy of the flow field is shown theoretically and numerically; computational models such as the eddy-damped quasi-normal Markovian (EDQNM) method by Naulin,[33] in which satiation of the zonal flow growth is studied; and shell models by Gürcan *et al.*,[41] in which the form of the fluctuation spectrum in a drift wave-zonal flow system is found. All of these focus on the role of non-local interactions in a closure approach.

A. Modulational instability

We first study the modulational instability of a wave spectrum to a seed zonal flow $\delta\langle v_x\rangle$ (with wave number $\boldsymbol{q}$ and eigen-frequency $\Omega_{\boldsymbol{q}}$). For a slowly varying, large scale shear flow, N_k changes adiabatically with the shear flow perturbation, and the modulational response of Rossby waves by the seed flow is determined by the linearized wave kinetic equation

$$\frac{\partial \widetilde{N}_k}{\partial t} + v_{gy}\frac{\partial}{\partial y}\widetilde{N}_k + \delta\omega_k \widetilde{N}_k = \frac{\partial(k_x\delta\langle v_x\rangle)}{\partial y}\frac{\partial N_0}{\partial k_y}, \tag{35}$$

where v_g is the group velocity of wave-packets and $\delta\omega_k$ represents nonlinear self-decorrelation rate via wave-wave interaction. For small perturbations $(\widetilde{N}_k, \delta\langle v_x\rangle) \sim e^{-i\Omega_{\boldsymbol{q}}t + iq_y y}$, the modulation of $\widetilde{N}_k$ becomes

$$\widetilde{N}_k = -iq\delta\langle v_x\rangle \frac{k_x}{-i(\Omega_{\boldsymbol{q}} - q_y v_{gy}) + \delta\omega_k}\frac{\partial N_0}{\partial k_y}, \tag{36}$$

so the growth rate of the seed zonal flow is given by

$$\gamma_q = -q^2\int d^2k \frac{k_x^2 k_y}{k^4}\frac{|\delta\omega_k|}{(\Omega_{\boldsymbol{q}} - q_y v_{gy})^2 + \delta\omega_k^2}\left(\frac{\partial N_0}{\partial k_y}\right), \tag{37}$$

where N_0 is normalized to the mean enstrophy density. The condition to have instability ($k_y\partial_{k_y}N_0 < 0$) is satisfied for most realistic equilibrium spectra for Rossby wave and drift wave turbulence. The fundamental mechanism of zonal flow generation includes not only local wave-wave interactions (in wavenumber space) but also non-local couplings between waves and flows. Therefore, zonal flow growth rate should depend on both the spectral structure of turbulence and properties of zonal flow itself. Equation (37) shows that the growth rate is indeed a function of wave spectrum $N_0(\boldsymbol{k})$ and zonal flow width q_y.

In the limit of $q_y v_{gy} \ll \delta\omega_k$, the response function is expanded as

$$\frac{|\delta\omega_k|}{(\Omega_{\boldsymbol{q}} - q_y v_{gy})^2 + \delta\omega_k^2} \approx \frac{1}{|\delta\omega|}\left(1 - \frac{q_y^2}{q_c^2}\right), \tag{38}$$

where the critical excursion length of wave-packets q_c^{-1} is defined as

$$q_c \equiv \frac{|\delta\omega_k|}{v_{gy}}. \tag{39}$$

The critical length can be understood as the mean free path of the wave-packets, considering wave-packets as quasi-particles. Thus, the $q_y v_{gy} \ll |\delta\omega_k|$ limit means that the width of the zonal flow is larger than the (cross-flow) mean free path of the wave-packets. This corresponds to the scale separation criterion of the wave kinetic equation. This expansion finally implies that the shear flow evolution consists of two parts and so Eq. (37) becomes

$$\gamma_{\boldsymbol{q}} = -q_y^2 D_t - q_y^4 H_t, \tag{40}$$

where the turbulent viscosity D_t and hyper-viscosity H_t are given by

$$D_t = \int d^2k \frac{k_x^2}{|\delta\omega_k|k^4}\frac{k_y\partial N_0}{\partial k_y}, \tag{41}$$

and

$$H_t = -\int d^2k \left(\frac{v_{gy}}{\delta\omega_k}\right)^2 \frac{k_x^2}{|\delta\omega_k|k^4}\frac{k_y\partial N_0}{\partial k_y}. \tag{42}$$

Given that $k_y\partial_{k_y}N_0 < 0$, the modulational calculation yields both a negative turbulent viscosity, which contributes to zonal flow growth, and a positive turbulent hyper-viscosity, which accounts for the saturation mechanism of zonal flow growth. Note that the large-scale instability due to negative diffusivity and small-scale damping due to higher-order diffusion are like what we observe in the K-S equation. The negative viscosity follows from the zeroth order term in the expansion of the response function. Thus, the resonance interaction between zonal flow and wave-packets results in the growth of zonal flow at the expense of the wave energy. Figure 4 shows a cartoon of a wave-packet traveling through a zonal flow. When the width of the zonal flow is larger than critical wave excursion length (the criterion for the expansion), the negative viscosity dominates, i.e., there is a net transfer of energy from the wave-packet to the zonal flow, and so the energy of the wave-packet decreases (right panel of Fig. 4). In the limit when zonal flow scale approaches the critical scale, the negative viscosity and positive hyper-viscosity balance each other, i.e., the net exchange of energy between the wave-packet and zonal flow goes to zero, and so the energy of the wave-packet remains the same (left panel of Fig. 4). The PV flux consisting of an effective viscosity and a positive hyper-viscosity is consistent with the structure of PV flux derived using the non-perturbative minimum enstrophy relaxation model. The critical length q_c^{-1} is similar to the characteristic scale l_c defined in the minimum enstrophy model because both of them characterize some scale at which viscous and hyper-viscous terms balance each other. At larger scales, negative viscosity dominates so zonal flows can grow; at smaller scales, positive hyper-viscosity dominates so zonal flows are damped. The difference of q_c^{-1} and l_c is that q_c^{-1} is defined in the context of wave dynamics, while l_c is defined by mean profiles.

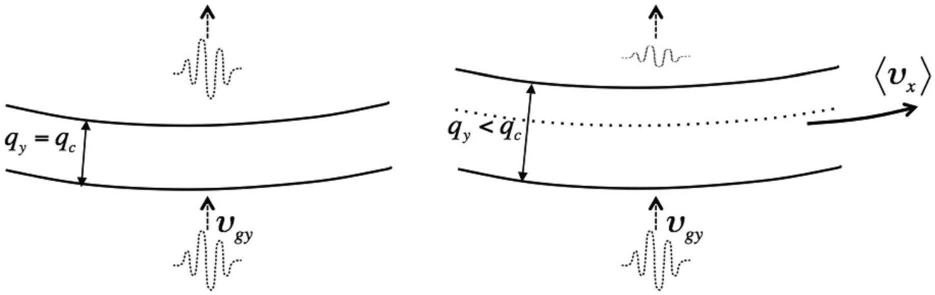

FIG. 4. Schematic illustration of a wave-packet traveling across a zonal flow. The intensity/energy of the wave-packet becomes weaker after crossing through a zonal flow with its width larger than the critical excursion length of the wave-packet (right). When the width of the zonal flow is equal to (or smaller than) the critical length, the energy of the wave-packet does not change (left).

B. Parametric instability

The modulational instability discussed in Subsection III A is developed for the wave kinetic limit, i.e., it applies to a broad wave spectrum, or, more specifically, the condition of $\delta\omega_k > \gamma_q$. In this subsection, we consider the case of a narrow turbulence spectrum. The instability of a gas of drift/Rossby waves to a seed zonal flow becomes the coherent hydrodynamic type (called parametric instability[7]), when the frequency spread of the wave spectrum is small compared to the growth rate of the zonal flow ($\delta\omega_k < \gamma_q$). Note that modulational instability and parametric instability are two regimes of the same process, namely, that small scale wave-packets are unstable with respect to a large scale perturbation. The dispersal of the wave-packets is what differentiates the regimes. The wave-packets which disperse slower than the growth rate of the perturbation react coherently to the perturbation, and so the instability belongs to the hydrodynamic regime, while the wave-packets which disperse faster than the growth rate of the perturbation belong to the kinetic regime.

A narrow spectrum of wave-packets in wave kinetic theory acts like pseudo-particles by analogy with charged particles in plasma fluids. The analogy between the pseudo-fluid and the plasma fluid is summarized in Table I. Note that one distinction between them is that there is no direct proportionality between the velocity $\boldsymbol{V}^w$ and momentum $\boldsymbol{P}^w$ of pseudo-fluid. The dynamics of the pseudo-fluid is derived from multiplying the wave kinetic equation, by v_{gy} and integrating over k

$$\frac{\partial}{\partial t}\int v_{gy}N_k dk + \int v_{gy}\frac{\partial}{\partial y}v_{gy}N_k dk - \int k_x\delta\langle v_x\rangle'\frac{\partial}{\partial k_y}v_{gy}N_k = -\int v_{gy}\delta\omega_k N_k dk, \tag{43}$$

TABLE I. Analogy between pseudo-fluid and plasma fluid.

	Pseudo-fluid	Plasma fluid
Elements	Wave-packets	Charged particles (species α)
Distribution function	$N_{\boldsymbol{k}}(\boldsymbol{k},\omega_{\boldsymbol{k}})$	$f_\alpha(\boldsymbol{r},\boldsymbol{v},t)$
Mean free path	$\lvert\boldsymbol{v}_g\rvert/\delta\omega_{\boldsymbol{k}}$	$1/n_\alpha\sigma$
Density	$n^w = \int N_{\boldsymbol{k}}d\boldsymbol{k}$	$n_\alpha = \int f_\alpha d\boldsymbol{v}$
Momentum	$\boldsymbol{P}^w = \int \boldsymbol{k}N_{\boldsymbol{k}}d\boldsymbol{k}$	$\boldsymbol{p}_\alpha = \int m_\alpha \boldsymbol{v} f_\alpha d\boldsymbol{v}$
Velocity	$\boldsymbol{V}^w = \dfrac{\int \boldsymbol{v}_g N_{\boldsymbol{k}}d\boldsymbol{k}}{\int N_{\boldsymbol{k}}d\boldsymbol{k}}$	$\boldsymbol{u}_\alpha = \dfrac{\int \boldsymbol{v}f_\alpha d\boldsymbol{v}}{\int f_\alpha d\boldsymbol{v}} = \dfrac{\boldsymbol{p}_\alpha}{m_\alpha n_\alpha}$

where $\delta\langle v_x\rangle' \equiv \partial_y\delta\langle v_x\rangle$. Normalizing Eq. (43) by the pseudo-density n^w, the first term on the left hand side gives $\partial_t V^w$, the velocity evolution of the pseudo-fluid. The second term can be decomposed into two parts

$$\frac{\int v_{gy}\frac{\partial}{\partial y}v_{gy}N_k dk}{\int N_k dk} = \int V_y^w\frac{\partial}{\partial y}V_y^w N_k dk + \frac{\int\left(v_{gy}-V_y^w\right)\frac{\partial}{\partial y}\left(v_{gy}-V_y^w\right)N_k dk}{\int N_k dk}. \tag{44}$$

The second part can be viewed as the "pressure" gradient of the pseudo-fluid. When the spectrum of wave-packets is narrow, the effective "temperature" is low because of weak dispersion. Thus, we can neglect this pressure term and consider a pure fluid type instability. The third term on the left hand side of Eq. (43) normalized by n^w is given by

$$\frac{-\int k_x\delta\langle v_x\rangle'\frac{\partial}{\partial k_y}v_{gy}N_k}{\int N_k dk} = -a\delta\langle v_x\rangle', \tag{45}$$

where

$$a = \frac{\int\left(\frac{2\beta k_x^2}{k^4} - \frac{8\beta k_x^2k_y^2}{k^6}\right)N_k dk}{\int N_k dk}. \tag{46}$$

The term on the right hand side of Eq. (43) is related to the dispersal of wave-packets due to wave-wave interaction, and so is neglected in the hydrodynamic regime for simplicity. Finally, putting the four terms together, we obtain the dynamic equation for the pseudo-fluid

$$\frac{\partial}{\partial t}V_y^w + V_y^w\frac{\partial}{\partial y}V_y^w = -a\delta\langle v_x\rangle'. \tag{47}$$

This equation is equivalent to the inviscid Burgers' equation with a source term on the right hand side contributed by the zonal shear. Given that the development of discontinuities (shock waves) is a crucial phenomenon that arises with the Burgers' equation, we speculate that coherent structures/shocks composed of wave-packets may form due to zonal shear stirring. It is worth noting that while the shock structure is the result of the nonlinear term, the zonal flow is what triggers it. Interestingly, shock formation in spectral-space, on account of zonal flow shearing, has been found in the study of intermittency in drift wave-zonal flow turbulence by Diamond and Malkov.[42] Using basic conservation and symmetry properties, they derive a generalized Burgers' equation for the wave action density in the radial wave number (i.e., k_y in this paper) space, in that damping and spatial propagation terms are present. The shock solutions to the generalized Burgers' equation are events of spectral pulses, which correspond to wave-packets, propagate ballistically to higher k_y. The transport in k_y results from shearing-induced refraction.

To relate the dynamics of the pseudo-fluid dynamics to zonal flow generation, we write the Reynolds stress in terms of the wave action density and the Rossby wave group velocity: $\langle \tilde{v}_y \tilde{v}_x \rangle = \int v_{gy} k_x N_k d^2k$. In the fluid limit, the right hand side is approximated as $V_y^w P_x^w$, which is the pseudo-momentum flux carried by the pseudo-fluid. The evolution of the zonal flow then becomes

$$\frac{\partial}{\partial t}\delta\langle v_x \rangle = -\frac{\partial}{\partial y} V_y^w P_x^w. \tag{48}$$

Next, we consider the instability to small perturbations by taking $V_y^w = V_{y,0}^w + \widetilde{V_y^w}$; $P_x^w = P_{x,0}^w + \widetilde{P_x^w}$ with the perturbations linearly proportional to the zonal shear $\delta\langle v_x \rangle'$. We can derive two coupled equations for the perturbations from Eqs. (47) and (48)

$$\frac{\partial}{\partial t}\widetilde{V_y^w} + V_{y,0}^w \frac{\partial}{\partial y}\widetilde{V_y^w} = -a\delta\langle v_x \rangle', \tag{49}$$

$$\frac{\partial}{\partial t}\delta\langle v_x \rangle = -V_{y,0}^w \frac{\partial}{\partial y}\widetilde{P_x^w} - P_{x,0}^w \frac{\partial}{\partial y}\widetilde{V_y^w}. \tag{50}$$

The pseudo-momentum P_x^w evolves in the same way as the zonal flow, which is shown from multiplying the wave kinetic equation by k_x and integrating over $\boldsymbol{k}$

$$\frac{\partial}{\partial t}\int k_x N_x d^2k = -\frac{\partial}{\partial y}\int v_{gy} k_x N_k d^2k. \tag{51}$$

Because only $\widetilde{P_x^w}$ contributes to the time derivative term, we substitute $\widetilde{P_x^w}$ with $\delta\langle v_x \rangle$ in Eq. (50) and rewrite the coupled equations in the Fourier form

$$(-i\Omega_q + iqV_{y,0}^w)\widetilde{V_y^w} = -iqa\delta\langle v_x \rangle, \tag{52}$$

$$(-i\Omega_q + iqV_{y,0}^w)\delta\langle v_x \rangle = -iqP_{x,0}^w \widetilde{V_y^w}, \tag{53}$$

so that we obtain the dispersion relation

$$(\Omega_q - qV_{y,0}^w)^2 = q^2 a P_{x,0}^w. \tag{54}$$

In monochromatic limit, a and $P_{x,0}^w$ are simply

$$a = \frac{2\beta k_x^2}{k^4}\left(1 - \frac{4k_y^2}{k^2}\right); \quad P_{x,0}^w = -\frac{k^4 |\psi_k|^2}{2\beta}, \tag{55}$$

and the reduced dispersion relation

$$(\Omega_q - qV_{y,0}^w)^2 = -q^2 k_x^2 |\psi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right), \tag{56}$$

is consistent with the result obtained by Smolyakov *et al.*[7] The coherent instability requires the condition of $k_x^2 - 3k_y^2 > 0$, i.e., radially extended, anisotropic turbulence. In other words, the wave structure set the marginality of the coherent secondary instability. Note that the criterion required for modulational instability does not depend on the wave spatial structure.

The growth rate of the zonal flow in Eq. (56) is proportional to the zonal flow wave number $|q|$, showing that PV transport in parametric instability is convective. This type of momentum transport may be faster than turbulent momentum diffusion. However, the saturation mechanism of this instability needs to be investigated further. The structure of the PV flux in the PV-avalanche relaxation (relaxation of a PV deviation back to a self-organized state) model contains a convective transport of PV. Here, we derive the transport coefficient for the convective PV transport in weak turbulence and narrow turbulence spectrum limit. From Eqs. (46) and (54), we can see that the transport coefficient depends on the fluctuation spectrum.

IV. CONCLUSION AND DISCUSSION

In this paper, we have explored different approaches to the calculation of the PV flux in quasi-2D turbulent systems which conserve PV. In the first part of this paper, we non-perturbatively deduced the general forms of PV flux from two relaxation models: (1) the minimum enstrophy relaxation model using selective decay principle and (2) the PV-avalanche model using the joint reflection symmetry principle. The structure of PV flux derived from both relaxation models consists of a viscous and a hyper-viscous transport of PV. The PV flux deduced from the PV-avalanche model has another convective term, which is, however, dependent on the gradient of the mean PV profile. In the second part of the paper, we calculated the transport coefficients using perturbation theory. A negative viscosity and a positive hyper-viscosity are derived from the broad-band modulational analysis, while a coefficient associated with the convective term is obtained from the narrow-band parametric analysis. The results of this paper are listed below and summarized in Table II.

In the minimum enstrophy relaxation model, we asked what form must the mean field PV flux have so as to dissipate enstrophy while conserving kinetic energy. The nonlinear term is annihilated in the end state of selective decay. We derived PV flux by writing the nonlinear term as N and $\partial_y \Gamma_q$. We showed that the results of these two formulations are consistent. The finding from the $\partial_y \Gamma_q$ approach is considered the primary result because the $\partial_y \Gamma_q$ formation is more accurate and better satisfies the mean field approximation. The PV flux is shown to be $\Gamma_q = \langle v_x \rangle^{-1} \partial_y[\mu \partial_y(\langle v_x \rangle^{-1} \partial_y \langle q \rangle)]$; it consists of diffusive and hyper-diffusive terms. We noted that there are other forms of PV flux which can minimize enstrophy while conserving energy. In this work, we studied only the simplest, smoothest form of the PV flux. We showed that the hyper-viscosity is positive and that the hyper-viscous term damps the energy of the mean zonal flow. Thus, the hyper-viscosity reflects the

TABLE II. Elements of the PV flux from structural, non-perturbative approaches, and perturbative analyses.

	PV flux	Convective	Viscous	Hyper-viscous	Coefficients
(Non-perturbative)	Min. enstrophy relaxation		•	•	
	PV-avalanche relaxation	•	•	•	
(Perturbative)	Modulational instability		•	•	$D_t(<0), H_t(>0)$
	Parametric instability	•			$\gamma_q(\sim \lvert q \rvert)$

saturation mechanism of zonal flows and the scale dependence of the momentum flux. The results are pragmatically useful in the context of transport modeling, where the problems of zonal flow scale and saturation are important.

We found the homogenized quantity in the relaxed state to be the ratio of PV gradient to zonal flow velocity, implying that strong zonal flows are located at sharp PV gradients. The observation that the structure of the relaxed state is consistent with the structure of the PV staircase suggests that the staircase arises naturally as a consequence of minimum enstrophy relaxation and links inhomogeneous PV mixing to minimum enstrophy relaxation. We demonstrated that turbulence spreading is linked to PV mixing by showing the dependence of energy flux on PV flux: $\Gamma_E = -\int \Gamma_q \langle v_x \rangle dy = \mu \partial_y (\langle v_x \rangle^{-1} \partial_y \langle q \rangle)$. Since the spreading flux is driven by the gradient of the quantity which ultimately is homogenized, it vanishes in the relaxed state. A relaxation rate was derived using linear perturbation theory. We found the "minimum enstrophy" required to sustain a zonal flow of a certain level in the relaxed state satisfies: $q_m^2 > 3\lambda \langle v_x \rangle^2 / 8$. A characteristic scale l_c was defined from the homogenized quantity, $l_c = |\partial_y \langle q \rangle / \langle v_x \rangle|^{-1/2}$, so that positive hyper-viscosity dominates at scales smaller than l_c, while effective viscosity dominates at scales larger than l_c. We noted that l_c is similar to the Rhines scale. Rhines scale and l_c become indistinguishable when $\widetilde{v}_{rms}$ and $\langle v_x \rangle$ are comparable.

In the PV-avalanche model, the form of the PV flux, which is driven by the deviation from the self-organized state, is constrained by the joint reflection symmetry condition. We found that one of the simplest forms of the PV flux contains a diffusive term, a hyper-diffusive term, and a nonlinear convective term. The PV equation has the same structure as the Kuramoto-Sivashinsky equation, which is known for its negative diffusion and higher-order stabilizing dissipation. The structure of viscous and hyper-viscous transport of PV was shown previously in the minimum enstrophy model, while the coefficients of viscosity and hyper-viscosity were derived later for the modulational instability calculation. The convective transport of PV, which suggests intermittent PV transport during turbulence self-organization, was not explicitly shown in the selective decay analysis. Nevertheless, we noted that for the case of avalanche-like transport, δq is counted as the deviation of the local gradient from the mean (critical) gradient, and the transport coefficients, constrained by the presence of the mean gradient, will depend on δq. Thus, a nonlinear convective component of PV flux ($\Gamma[\delta q] \sim \delta q^2$) is equivalent to a generalized diffusive transport (i.e., $\Gamma_q \sim -D(\nabla q - \nabla q_{\rm crit})\nabla q \to -D(\delta q)\delta q$, with $D(\delta q) \to D_0 \delta q$). These may represent similar transport processes.

To sum up, we compare the structures of PV flux derived via two relaxation models: the transport flux-oriented mean field theory (based on the minimum enstrophy principle) and the generalized Fickian mean field theory (based on the joint reflection symmetry principle); the PV flux of the later includes the terms of the former together with a convective term. We proposed that a profile-dependent transport coefficient gives a more general form of the PV flux of systems where negative viscosity phenomena take place, since the relative dependence on the instant PV profile, especially the profile gradient, is crucial to the ballistic propagation of PV defects—avalanches. While noting that the profile gradient dependence of the transport coefficients has potential effects on transport behavior, the detail of the dependency is beyond the scope of this work.

To derive the transport coefficients rigorously, we then used perturbation theory to study the instability of an ensemble of wave packets to a large scale seed perturbation. Instabilities of two regimes were considered: (1) modulational instability for a broad turbulence spectrum and (2) parametric instability for a narrow turbulence spectrum. In the modulational instability analysis of the wave kinetic equation, we found that to the lowest order, PV flux is composed of a negative viscous and positive hyper-viscous terms. The viscous and hyper-viscous transport of PV is shown in the minimum enstrophy relaxation model and the PV-avalanche model as well. The negative viscosity from the resonance interaction between zonal flow and wave-packets contributes to the growth of the zonal flow. The positive hyper-viscosity reflects the saturation mechanism of zonal flows and the scale dependence of the PV flux. A critical scale q_c^{-1} was defined so that negative viscosity term is dominant at scales larger than q_c^{-1}, while positive hyper-viscosity term is dominant at scales smaller than q_c^{-1}. In the parametric instability analysis, we derived a model of a pseudo-fluid composed of wave-packets. The dynamic equation for the pseudo-fluid is the inviscid Burgers' equation, with a source term contributed by the zonal flow. This suggests that coherent structures—wave-packets—may form due to zonal shear stirring. The PV transport associated with the pseudo-fluid is a convective process, since the growth rate of the zonal flow in the parametric instability is proportional to the zonal flow wave number. The PV transport coefficients were shown to depend on the fluctuation spectrum.

PV mixing, the fundamental process for zonal flow generation, is directly linked to the forward enstrophy cascade in wave-number space. The importance of such small scale mixing processes is seen from the appearance of hyper-viscosity in the PV flux, which contributes to zonal flow energy damping. The terms in the PV flux which contribute to zonal flow energy growth (i.e., effective negative viscosity,) however, are not well reconciled with the picture of diffusive mixing of PV in real space. Here, we offer a possible explanation, based on the

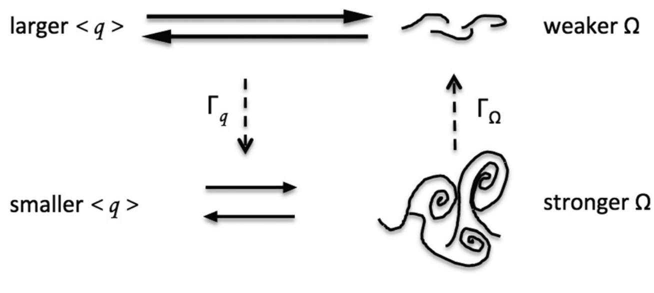

FIG. 5. PV mixing tends to transport PV from the region of larger mean PV (i.e., stronger zonal shears) to the region of smaller mean PV, while turbulence spreading tends to transport turbulence from the region of stronger turbulent intensity (turbulent potential enstrophy $\Omega \equiv \langle \tilde{q}^2 \rangle$ or turbulent energy $\langle \tilde{v}^2 \rangle$) to the region of weaker turbulent intensity.

connection between PV mixing and turbulence spreading derived from the minimum enstrophy analyses. We may consider turbulence spreading as a process which contributes to up-gradient, or "anti-diffusive," mixing of PV. The argument is as follows: it is reasonable to assume that PV mixing in real space tends to transport PV from the region of larger mean PV to the region of smaller mean PV. Because a stronger mean vorticity corresponds to a stronger shearing field which suppresses turbulence, the PV mixing process tends to transport PV away from the region of weak (turbulence) excitation toward the region of stronger excitation. In contrast, the spreading of turbulent enstrophy tends to transport enstrophy from the strongly turbulent region to the weakly turbulent region (Fig. 5). When the tendency of turbulence spreading is greater, the net transport of PV appears up-gradient, and so the apparent effective viscosity becomes negative. The relaxed state is reached when PV mixing and turbulent enstrophy spreading are balanced. The total PV fluxes we calculated in the relaxation model and the modulational instability analysis include both trends.

We conclude by noting that, the dynamics of PV flux derived analytically in this work has not been confirmed by numerical tests. Therefore, important topics for future research would be developing numerical simulation tests of the models discussed in this paper and comparing the results to the analytical predictions. Some of the numerical tests we can do include: (i) examining the form of the PV flux during relaxation and testing the hyper-viscous model derived in this paper, (ii) calculating the profile of the quantity $\partial_y \langle q \rangle / \langle v_x \rangle$, predicted to be homogenized in the relaxed state, (iii) calculating the minimum enstrophy (q_m^2) in the relaxed state, and (iv) studying the PV flux spectrum, especially low frequency components, with avalanching in mind.

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